Bosonic Reduction of Susy Generalized Harry Dym Equation

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Abstract

In this paper we construct the two component supersymmetric generalized Harry Dym equation which is integrable and study various properties of this model in the bosonic limit. In particular, in the bosonic limit we obtain a new integrable system which, under a hodograph transformation, reduces to a coupled three component system. We show how the Hamiltonian structure transforms under a hodograph transformation and study the properties of the model under a further reduction to a two component system. We find a third Hamiltonian structure for this system (which has been shown earlier to be a bi-Hamiltonian system) making this a genuinely tri-Hamiltonian system. The connection of this system to the modified dispersive water wave equation is clarified. We also study various properties in the dispersionless limit of our model.

1 Introduction

The Harry Dym equation [1, 2] is an important dynamical system which is integrable and finds applications in several physical systems. Together with the Hunter-Zheng equation [3, 4, 5, 6, 7], they define a hierarchy of integrable systems for both positive and negative integers. This system has been vigorously studied in the past as well as more recently from various points of view. In particular, the supersymmetrization of such a system has been discused systematically in [8, 9] and a two component generalized Harry Dym equation has been constructed in [10].

The construction of supersymmetric integrable systems and the understanding of their properties is important for a variety of reasons [11, 12, 13]. One of the interesting features lies in the fact that supersymmetric integrable models can lead to new integrable systems in the bosonic limit [13, 14, 15]. As we have already pointed out in connection with the $N = 2$ supersymmetrization of the Harry Dym equation, one of the supersymmetrizations leads to an interesting new coupled integrable system in the bosonic limit [8]. It is with this aim that we have chosen to construct and study the two component supersymmetric (susy) generalized Harry Dym equation in this paper. The construction is quite interesting and leads to many new features including the fact that in the bosonic limit we obtain new integrable equations.

Our paper is organized as follows. In section **2**, we recapitulate briefly the two component generalized Harry Dym equation and some of its properties. In section **3**, we construct the two component supersymmetric Harry Dym equation which is integrable. The Lax representation as well as the Hamiltonian structures are discussed in detail in terms of different variables. We construct the hodograph transformation for the bosonic limit of such a system. In section **4**, we address the question of how a Hamiltonian structure transforms under a hodograph transformation. Using this, we derive the Hamiltonian structure for the transformed equation and show that under some approximation, the system reduces to a three component coupled MKdV system of equations. A further reduction, in section **5** takes this equation to one which has been studied earlier and we find a new Hamiltonian structure for this system which genuinely makes it a tri-Hamiltonian system. The connection of this system with the modified dispersive water wave equation is also clarified. In section **6**, the dispersionless limit of our system of equations is studied systematically and various associated properties including polynomial and nonpolynomial charges are derived. We conclude with a

brief summary in section **7**.

2 Harry Dym and the Generalized Harry Dym Equations:

As is well known, the Harry Dym equation [1] can be written in the form

$$\frac{\partial w}{\partial t} = w^3 w_{xxx}, \tag{1}$$

where subscripts denote derivatives with respect to the corresponding variables and the equation can be obtained from the following Lax representation

$$\frac{\partial L}{\partial t} = 4\Big[(L)_{\geq 2}^{\frac{2n+1}{2}}, L\Big], \tag{2}$$

with $n = 1$ and

$$L = w^2 \partial^2. \tag{3}$$

Here ≥ 2 refers to the projection onto the sub-algebra of the pseudo differential operator P, namely,

$$(P)_{\geq 2} = \sum_{i=2}^{\infty} a_i \partial^i. \tag{4}$$

The hierarchy of Harry Dym equations is integrable and finds applications in various physical examples.

The Harry Dym equation can be generalized to two dynamical variables in the following way [10]. Let us consider a Lax operator in the product form

$$L = w^2 \partial^2 u^2 \partial^2, \tag{5}$$

where w, u denote two dynamical variables. It is straightforward to check that the non-standard Lax representation

$$\frac{\partial L}{\partial t} = 4\Big[(L)_{\geq 2}^{3/4}, L\Big], \tag{6}$$

leads to the system of coupled equations for w and u of the form

$$\begin{aligned} \frac{\partial w}{\partial t} &= w^3 \Big(w^{-1/2} u^{3/2}\Big)_{xxx} \\ \frac{\partial u}{\partial t} &= u^3 \Big(u^{-1/2} w^{3/2}\Big)_{xxx}. \end{aligned} \tag{7}$$

This is the two component generalized Harry Dym equation which can be written in the Hamiltonian form as

$$\begin{pmatrix} w \\ u \end{pmatrix}_t = \begin{pmatrix} 0 & w^3\partial^3 u^3 \\ u^3\partial^3 w^3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta u} \end{pmatrix}, \tag{8}$$

where

$$H = -2\int \mathrm{d}x \left(wu\right)^{-\frac{1}{2}}. \tag{9}$$

We note that we can rewrite the system of equations (7) in a simpler form in terms of the new variables

$$u = ae^{b} \quad , \quad w = ae^{-b}, \tag{10}$$

as

$$\begin{aligned} a_t &= a^3(a_{xxx} + 12a_x b_x^2 + 12b_{xx}b_x a), \\ b_t &= -2a^2((ab)_{xxx} - a_{xxx}b + 4b_x^3 a). \end{aligned} \tag{11}$$

This equation can be writen in the Hamiltonian form

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = \frac{1}{4} \begin{pmatrix} a^3\left(X - X^\dagger\right)a^3 & -a^3\left(X + X^\dagger\right)a^2 \\ a^2\left(X + X^\dagger\right)a^3 & -a^2\left(X - X^\dagger\right)a^2 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix}, \tag{12}$$

where the operator X can be identified with

$$X = e^{2b}\partial^3 e^{-2b}, \tag{13}$$

with $X^\dagger$ representing the Hermitian conjugate and

$$H = -2\int \mathrm{d}x\, a^{-1}. \tag{14}$$

3 Supersymmetric Generalized Harry Dym Equation:

The supersymmetric extensions of the Harry Dym equation have already been constructed in [8]. Let us briefly recapitulate some of the features of the $N = 2$ extended supersymmetric Harry Dym equation which is most useful

from the point of view of constructing the generalized supersymmetric Harry Dym equation. In this case the bosonic dynamical variable is generalized to a bosonic $N = 2$ superfield

$$W = w_0 + \theta_1 \chi_1 + \theta_2 \chi_2 + \theta_1 \theta_2 w_1, \tag{15}$$

where w_1 is the original bosonic dynamical variable while w_0 is the new bosonic dynamical variable and $\chi_i, i = 1, 2$ are the new fermionic dynamical variables necessary for $N = 2$ supersymmetry. We note that $\theta_i, i = 1, 2$ represent the two Grassmann coordinates of the $N = 2$ superspace. We can define the two supercovariant derivatives on this space as

$$D_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial x} \quad , \quad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \frac{\partial}{\partial x}, \tag{16}$$

which satisfy

$$\{D_1, D_2\} = 0, \quad D_1^2 = D_2^2 = \partial. \tag{17}$$

In [8] we showed that it is possible to construct four different $N = 2$ supersymmetric Lax operators that lead to consistent equations. Namely, the general Lax operator with the $N = 2$ superfield T

$$\begin{aligned} L &= T^{-1}\partial^2 + k_1(D_1 T^{-1})\partial + k_1(D_2 T^{-1})\partial + \\ &\quad (k_2(D_1 D_2 T)T^{-2} + k_3(D_1 T)(D_2 T)T^{-3})D_1 D_2, \end{aligned} \tag{18}$$

leads to a consistent non-standard Lax representation

$$\frac{\partial L}{\partial t} = 4\Big[(L)^{\frac{3}{2}}_{\geq 2}, L\Big], \tag{19}$$

only for $k_1 = k_2 = k_3 = 0$ or $k_1 = k_2 = -\frac{k_3}{2} = 1$ or $k_1 = \frac{1}{2}, k_2 = k_3 = 0$ or $k_1 = k_2 = \frac{1}{2}, k_3 = \frac{3}{4}$. We will consider here only the last case which allows us to carry out the construction of the two component supersymmetric generalized Harry Dym equation.

Let us note (which has been pointed out in [8] as well) that in this particular case, the Lax operator (18) can be written as

$$L = -(W D_1 D_2)^2, \tag{20}$$

where W is related to T and the Lax equation (19) leads to the dynamical equations

$$\begin{aligned} W_t &= \frac{1}{8}\Big(6(D_1 D_2 W)(D_1 D_2 W_x)W^2 - 2W_{xxx}W^3 \\ &\quad +3(D_1 W_{xx})(D_1 W)W^2 + 3(D_2 W_{xx})(D_2 W)W^2\Big). \end{aligned} \tag{21}$$

This equation can be written in the Hamiltonian form

$$W_t = \mathcal{D}_1 \frac{\delta H_0}{\delta W}, \tag{22}$$

where

$$\mathcal{D}_1 = -\frac{1}{2} W^2 D_1 D_2 \partial W^2, \quad H_0 = -\frac{1}{4} \int \mathrm{d}x \, \mathrm{d}^2\theta \, (D_2 W)(D_1 W) W^{-1}, \tag{23}$$

with $\mathrm{d}^2\theta = \mathrm{d}\theta_1 \, \mathrm{d}\theta_2$.

In order to construct the two component supersymmetric generalized Harry Dym equation, let us consider the two differential operators on the $N = 2$ superspace

$$L_1 = F D_1 D_2, \quad L_2 = G D_1 D_2, \tag{24}$$

each of which has the form of the square root of (20). Here F, G denote two bosonic superfields on the $N = 2$ superspace. Let us next construct a new Lax operator as the product

$$L = -L_1 L_2. \tag{25}$$

It can be checked that the two component susy generalized Harry Dym equation follows from the non-standard Lax representation

$$\frac{\partial L}{\partial t} = 4[L^{\frac{3}{2}}_{\geq 2}, L] \tag{26}$$

In fact, the equations are derived in a much simpler manner with a parameterization of the form (10), namely,

$$F = e^W U, \quad G = e^{-W} U. \tag{27}$$

With this parametrization, the dynamical equations can be written in the Hamiltonian form

$$\begin{pmatrix} U \\ W \end{pmatrix}_t = \frac{1}{4} \begin{pmatrix} U^2 \left(X - X^\dagger\right) U^2 & -U^2 \left(X + X^\dagger\right) U \\ U \left(X + X^\dagger\right) U^2 & -U \left(X - X^\dagger\right) U \end{pmatrix} \begin{pmatrix} \frac{\delta H_2}{\delta U} \\ \frac{\delta H_2}{\delta W} \end{pmatrix}, \tag{28}$$

where $X = e^W D_1 D_2 \partial e^{-W}$ and

$$H_2 = \mathrm{Tr}\left(L^{\frac{1}{2}}\right) = \int \mathrm{d}x \, \mathrm{d}^2\theta \left[(D_2 W)(D_1 W) U - \frac{(D_2 U)(D_1 U)}{U}\right]. \tag{29}$$

It is also possible to write these equations in a still simpler form by introducing the variables

$$U = \frac{1}{\sqrt{fg}}, \quad W = \frac{1}{2}\left(\ln g - \ln f\right), \tag{30}$$

in terms of which we have

$$\begin{pmatrix} f \\ g \end{pmatrix}_t = \begin{pmatrix} 0 & Y \\ -Y^\dagger & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_2}{\delta f} \\ \frac{\delta H_2}{\delta g} \end{pmatrix}, \tag{31}$$

where

$$Y = D_1 D_2 \partial, \quad H_2 = -4 \int \mathrm{d}x\, \mathrm{d}^2\theta\, \frac{1}{\sqrt{f}}\, D_1 D_2 \left(\frac{1}{\sqrt{g}}\right).$$

It is now obvious that the Hamiltonian structure in (31) satisfies Jacobi identity. We will, however, continue to work with the first representation of the Hamiltonian equations (28). This system of equations is integrable and the conserved charges can be obtained from the traces of the Lax operator in the standard manner.

In the bosonic limit where we can restrict the superfields and the Hamiltonian to the forms

$$\begin{aligned} W &= w_0 + \theta_1\theta_2 w_1, \quad U = u_0 + \theta_1\theta_2 u_1, \\ H_2 &= \int \mathrm{d}x \left(w_{0x}^2 u_0 + w_1^2 u_0 - \frac{u_{0x}^2 + u_1^2}{u_0}\right), \end{aligned} \tag{32}$$

equation (28) leads to a system of four interacting equations of the form

$$\begin{aligned} w_{0t} &= \frac{1}{2}\big(2w_{0xxx}u_0^3 + 6w_{0xx}u_{0x}u_0^2 - w_{0x}^3 u_0^3 + 3w_{0x}u_{0x}^2 u_0 \\ &\quad +3w_{0x}u_1^2 u_0 - 3w_1^2 w_{0x}u_0^3\big), \\ w_{1t} &= \frac{1}{2}\big(-6w_{0xx}u_{1x}u_0^2 - 3w_{0x}^3 u_1 u_0^2 - 6w_{0x}u_{1xx}u_0^2 - 12w_{0x}u_{1x}u_{0x}u_0 \\ &\quad +3w_{0x}u_1^3 - 6w_{0x}u_1 u_{0xx}u_0 + 3w_{0x}u_1 u_{0x}^2 + 2w_{1xxx}u_0^3 \\ &\quad +12w_{1xx}u_{0x}u_0^2 + 12w_{1x}u_{0xx}u_0^2 + 12w_{1x}u_{0x}^2 u_0 + 6w_{1x}w_{0x}^2 u_0^3 \\ &\quad +6w_{1x}w_1^2 u_0^3 + 6w_1^3 u_{0x}u_0^2 - 3w_1^2 w_{0x}u_1 u_0^2 + 6w_1 u_{0xxx}u_0^2 \\ &\quad +12w_1 u_{0xx}u_{0x}u_0 - 6w_1 u_{1x}u_1 u_0 + 6w_1 w_{0xx}w_{0x}u_0^3 + 12w_1 w_{0x}^2 u_{0x}u_0^2\big), \\ u_{0t} &= \big(u_{0xxx}u_0^3 - 3u_{1x}u_1 u_0^2 + 3w_{1x}w_1 u_0^4 + 3w_1^2 u_{0x}u_0^3\big), \end{aligned}$$

$$
\begin{aligned}
u_{1t} &= \frac{1}{2}\left(2u_{1xxx}u_0^3 + 6u_{1xx}u_{0x}u_0^2 + 6u_{1x}u_{0xx}u_0^2 - 12u_{1x}u_1^2u_0\right.\\
&\quad +6u_1u_{0xxx}u_0^2 + 6w_{0xx}w_{0x}u_1u_0^3 + 6w_{0x}^2u_{1x}u_0^3 + 6w_{0x}^2u_1u_{0x}u_0^2\\
&\quad -6w_{1xx}w_{0x}u_0^4 - 6w_{1x}w_{0xx}u_0^4 - 24w_{1x}w_{0x}u_{0x}u_0^3 + 12w_{1x}w_1u_1u_0^3\\
&\quad -3w_1^3w_{0x}u_0^4 + 12w_1^2u_1u_{0x}u_0^2 - 6w_1w_{0xx}u_{0x}u_0^3 - 3w_1w_{0x}^3u_0^4\\
&\quad \left.-12w_1w_{0x}u_{0xx}u_0^3 - 9w_1w_{0x}u_{0x}^2u_0^2 + 3w_1w_{0x}u_1^2u_0^2\right). \qquad (33)
\end{aligned}
$$

When $w_0 = w_1 = 0$, this system of equations reduces to the bosonic limit of the $N = 2$ susy Harry Dym equation (21). On the other hand, when we set $w_0 = u_1 = 0$, and identify $u_0 = a, w_1 = 2b_x$ the system of equations goes over to the generalized Harry Dym equation (11) (except for an overall factor in the second equation which we are unable to get rid of by any scaling).

Let us next study the behavior of this system of equations under a hodograph transformation [16] of the variables (x, t) to (y, τ) defined through

$$
x = p_0(y, \tau), \quad \tau = t, \qquad (34)
$$

so that we have

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \frac{1}{p_{0y}}\frac{\partial}{\partial y},\\
\frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} + p_{0\tau}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + \frac{p_{0\tau}}{p_{0y}}\frac{\partial}{\partial y}. \qquad (35)
\end{aligned}
$$

Defining the variables

$$
w_0 = q_0, \quad w_1 = q_1, \quad u_0 = p_{0y}, \quad u_1 = p_1, \qquad (36)
$$

the system of equations (33) goes over to

$$
\begin{aligned}
p_{0\tau} &= \frac{1}{2}\left(2p_{0yyy} - 3p_{0yy}^2p_{0y}^{-1} - 3p_1^2p_{0y} + 3q_1^2p_{0y}^3\right),\\
p_{1\tau} &= \frac{1}{2}\left(2p_{1yyy} + 6p_{1y}p_{0yyy}p_{0y}^{-1} - 9p_{1y}p_{0yy}^2p_{0y}^{-2} - 15p_{1y}p_1^2\right.\\
&\quad +6p_1p_{0yyyy}p_{0y}^{-1} - 24p_1p_{0yyy}p_{0yy}p_{0y}^{-2} + 18p_1p_{0yy}^3p_{0y}^{-3} + 6q_{0yy}q_{0y}p_1\\
&\quad +6q_{0y}^2p_{1y} - 6q_{1yy}q_{0y}p_{0y} - 6q_{1y}q_{0yy}p_{0y} - 12q_{1y}q_{0y}p_{0yy}\\
&\quad +12q_{1y}q_1p_1p_{0y}^2 - 3q_1^3q_{0y}p_{0y}^3 + 3q_1^2p_{1y}p_{0y}^2 + 12q_1^2p_1p_{0yy}p_{0y}\\
&\quad -6q_1q_{0yy}p_{0yy} - 3q_1q_{0y}^3p_{0y} - 12q_1q_{0y}p_{0yyy} + 9q_1q_{0y}p_{0yy}^2p_{0y}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& +3q_1q_{0y}p_1^2p_{0y}\big)\,, \\
q_{0\tau} &= \frac{1}{2}\left(2q_{0yyy}-q_{0y}^3\right), \\
q_{1\tau} &= \frac{1}{2}\big(-6q_{0yy}p_{1y}p_{0y}^{-1}-3q_{0y}^3p_1p_{0y}^{-1}-6q_{0y}p_{1yy}p_{0y}^{-1} \\
& +3q_{0y}p_1^3p_{0y}^{-1}-6q_{0y}p_1p_{0yyy}p_{0y}^{-2}+9q_{0y}p_1p_{0yy}^2p_{0y}^{-3}+2q_{1yyy} \\
& +6q_{1yy}p_{0yy}p_{0y}^{-1}+12q_{1y}p_{0yyy}p_{0y}^{-1}-9q_{1y}p_{0yy}^2p_{0y}^{-2}-3q_{1y}p_1^2 \\
& +6q_{1y}q_{0y}^2+9q_{1y}q_1^2p_{0y}^2+6q_1^3p_{0yy}p_{0y}-3q_1^2q_{0y}p_1p_{0y} \\
& +6q_1p_{0yyyy}p_{0y}^{-1}-12q_1p_{0yyy}p_{0yy}p_{0y}^{-2}+6q_1p_{0yy}^3p_{0y}^{-3}-6q_1p_{1y}p_1 \\
& +6q_1q_{0yy}q_{0y}+6q_1q_{0y}^2p_{0yy}p_{0y}^{-1}\big)\,. \qquad (37)
\end{aligned}
$$

We note from (37) that the equation for the variable q_0 is decoupled. As a result, we can set $q_0 = 0$ for simplicity. The Hamiltonian structure for the system of equation involving w_1, u_0, u_1 can be obtained from the Dirac reduction of the bosonic limit of the Hamiltonian operator in (28). In the simple case of two variables, for example, the Dirac reduction works as follows. Let us assume the equations of motion to have the form

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} P_{uu} & P_{uv} \\ P_{vu} & P_{vv} \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}, \qquad (38)
$$

where u, v denote the two dynamical variables with $P_{uu}, P_{uv}, P_{vu}, P_{vv}$ representing the elements of the Hamiltonian operator. If we can set $v = 0$, then it can be verified directly that [2, 17] the Hamiltonian structure reduces to

$$
u_t = \left(P_{uu} - P_{uv}P_{vv}^{-1}P_{vu}\right) \frac{\delta H}{\delta u}\bigg|_{v=0}, \qquad (39)
$$

for the reduced system.

For the three component system under study in (33) (with $w_0 = 0$), the Dirac reduction leads to

$$
\begin{pmatrix} u_0 \\ u_1 \\ w_1 \end{pmatrix}_t = \mathcal{D} \begin{pmatrix} \frac{\delta H_2}{\delta u_0} \\ \frac{\delta H_2}{\delta u_1} \\ \frac{\delta H_2}{\delta w_1} \end{pmatrix}, \qquad (40)
$$

where the matrix Hamiltonian operator $\mathcal{D}$ has the elements

$$
\mathcal{D}^{(1,1)} = \partial u_0^4 + u_0^4\partial,
$$

$$
\begin{aligned}
\mathcal{D}^{(1,2)} &= 4u_{1y}u_0^3 + 4u_1u_{0y}u_0^2 + 4u_1u_0^3\partial, \\
\mathcal{D}^{(1,3)} &= 2w_{1y}u_0^3 + 2w_1u_{0y}u_0^2 + 2w_1u_0^3\partial, \\
\mathcal{D}^{(2,2)} &= -\partial^3 u_0^4 - u_0^4\partial^3 - \partial(12u_{0y}^2u_0^2 - 4u_1^2u_0^2) - (12u_{0y}^2u_0^2 - 4u_1^2u_0^2)\partial, \\
\mathcal{D}^{(2,3)} &= 4w_{1y}u_1u_0^2 + 4w_1u_1u_{0y}u_0 + 4w_1u_1u_0^2\partial, \\
\mathcal{D}^{(3,3)} &= \partial^2 u_0^3 + u_0^2\partial^3 + \partial(w_1^2u_0^2 - 3u_{0y}^2) + (w_1^2u_0^2 - 3u_{0y}^2)\partial,
\end{aligned} \tag{41}
$$

and

$$
H_2 = \int \mathrm{d}x \left(w_1^2 u_0 - \frac{u_{0x}^2 + u_1^2}{u_0} \right). \tag{42}
$$

4 Transformation of the Hamiltonian Structure under the Hodograph Transformation:

In this section, we will discuss how the Hamiltonian structure transforms under a hodograph transformation. Namely, we already have the Hamiltonian structure for the three component equation in (33) with $w_0 = 0$. We would like to determine the Hamiltonian structure for the three component equation in (37) with $q_0 = 0$ which is obtained from (33) under a hodograph transformation. We note that the Hamiltonian (42) in terms of the three variables p_0, p_1 and q_1 takes the form

$$
H = \int \mathrm{d}y \left(-\frac{p_{0yy}}{p_{0y}^2} - p_1^2 + q_1^2 p_{0y}^2 \right). \tag{43}
$$

Furthermore, from the definition of the hodograph transformations in (35) and (36), we obtain

$$
\begin{pmatrix} u_0 \\ u_1 \\ w_1 \end{pmatrix}_t = S^{-1} \begin{pmatrix} p_0 \\ p_1 \\ q_1 \end{pmatrix}_\tau, \tag{44}
$$

where S has the form

$$
S = \begin{pmatrix} p_{0y}\partial^{-1}p_{0y}^{-1} & 0 & 0 \\ p_{1y}\partial^{-1}p_{0y}^{-1} & 1 & 0 \\ q_{1y}\partial^{-1}p_{0y}^{-1} & 0 & 1 \end{pmatrix}. \tag{45}
$$

The gradient of the Hamiltonian transforms under the hodograph transformation as

$$\begin{pmatrix} \frac{\delta H}{\delta u_0} \\ \frac{\delta H}{\delta u_1} \\ \frac{\delta H}{\delta w_1} \end{pmatrix} = K \begin{pmatrix} \frac{\delta H}{\delta p_0} \\ \frac{\delta H}{\delta p_1} \\ \frac{\delta H}{\delta q_1} \end{pmatrix}, \tag{46}$$

where

$$K = \begin{pmatrix} -p_{0y}^{-2}\partial^{-1}p_{0y} & -p_{0y}^{-2}\partial^{-1}p_{1y} & -p_{0y}^{-2}\partial^{-1}q_{1y} \\ 0 & p_{0y}^{-1} & 0 \\ 0 & 0 & p_{0y}^{-1} \end{pmatrix} = p_{0y}^{-1}S^{\dagger} \tag{47}$$

It follows now that

$$\begin{pmatrix} p_0 \\ p_1 \\ q_1 \end{pmatrix}_{\tau} = S\mathcal{D}\begin{pmatrix} \frac{\delta H_2}{\delta u_0} \\ \frac{\delta H_2}{\delta u_1} \\ \frac{\delta H_2}{\delta w_1} \end{pmatrix} = S\mathcal{D}K\begin{pmatrix} \frac{\delta H_2}{\delta p_0} \\ \frac{\delta H_2}{\delta p_1} \\ \frac{\delta H_2}{\delta q_1} \end{pmatrix} = \widetilde{\mathcal{D}}\begin{pmatrix} \frac{\delta H_2}{\delta p_0} \\ \frac{\delta H_2}{\delta p_1} \\ \frac{\delta H_2}{\delta q_1} \end{pmatrix}. \tag{48}$$

The form of the transformed Hamiltonian structure $\widetilde{\mathcal{D}}$ can now be determined easily to have the following elements

$$\begin{aligned}
\widetilde{\mathcal{D}}^{(1,1)} &= -2p_{0y}\partial^{-1}p_{0y}, \\
\widetilde{\mathcal{D}}^{(1,2)} &= -2p_{0y}\partial^{-1}p_{1y} + 4p_1p_{0y}, \\
\widetilde{\mathcal{D}}^{(1,3)} &= -2p_{0y}\partial^{-1}q_{1y} + 2q_1p_{0y}, \\
\widetilde{\mathcal{D}}^{(2,2)} &= -2(\partial p_{0yyy}p_{0y}^{-1} + p_{0yyy}p_{0y}^{-1}\partial) + 6(\partial p_{0yy}^2p_{0y}^{-2} + p_{0yy}^2p_{0y}^{-2}\partial) \\
&\quad -2p_{1y}\partial^{-1}p_{1y} + 4(\partial p_1^2 + p_1^2\partial) - 2\partial^3, \\
\widetilde{\mathcal{D}}^{(2,3)} &= -2p_{1y}\partial^{-1}q_{1y} + 2q_1p_{1y} + 4q_1p_1\partial), \\
\widetilde{\mathcal{D}}^{(3,3)} &= -2q_{1y}\partial^{-1}q_{1y} + \partial^2p_{0x}^{-2}\partial + \partial p_{0x}^{-2}\partial^2 + \partial q_1^2 + q_1^2\partial.
\end{aligned} \tag{49}$$

We can further simplify the equations (37) defining the new variables

$$f = \frac{p_{0yy}}{p_{0y}}, \quad g = p_1, \quad h = q_1p_{0y}, \tag{50}$$

which leads to the equations

$$f_\tau = \left(2f_{yy} - f^3 - 6g_yg - 3g^2f + 6h_yh + 3h^2f\right)_y,$$

$$\begin{aligned}
g_\tau &= \left(2g_{yy} - 5g^3 + 6gf_y - 3gf^2 + 3h^2g\right)_y + 6h_yhg,\\
h_\tau &= \left(2h_{yy} + 5h^3 + 6hf_y - 3hf^2 - 3hg^2\right)_y - 6hg_yg. \qquad (51)
\end{aligned}$$

The Hamiltonian structure $\widetilde{\mathcal{D}}$, in this case, transforms to

$$\begin{aligned}
\widetilde{\mathcal{D}}^{(1,1)} &= -2f_y\partial^{-1}f_y + \partial f^2 + f^2\partial - 2\partial^3,\\
\widetilde{\mathcal{D}}^{(1,2)} &= -2f_y\partial^{-1}g_y + 2\partial(\partial g + g\partial) + 2\partial gf + 2g\partial f,\\
\widetilde{\mathcal{D}}^{(1,3)} &= -2f_y\partial^{-1}h_y + 2\partial(\partial h + h\partial) + 2\partial hf + 2h\partial f,\\
\widetilde{\mathcal{D}}^{(2,2)} &= -2g_y\partial^{-1}g_y + \partial(f^2 + 4g^2 - 2f_y) + (f^2 + 4g^2 - 2f_y)\partial - 2\partial^3,\\
\widetilde{\mathcal{D}}^{(2,3)} &= -2g_y\partial^{-1}h_y + 4(\partial hg + hg\partial), \qquad (52)\\
\widetilde{\mathcal{D}}^{(3,3)} &= -2h_y\partial^{-1}h_y + \partial(4h^2 - f^2 + 2f_y) + (4h^2 - f^2 + 2f_y)\partial + 2\partial^3,
\end{aligned}$$

and the Hamiltonian in the new variables has the form

$$H = \int \mathrm{d}y\ \left(-f^2 - g^2 + h^2\right).$$

All the operators ∂ in the expression (52) refer to derivative operators with respect to the variable y. We note that $H_0 = \int \mathrm{d}y\, f$ is a conserved quantity which defines the Casimir of the Hamiltonian structure $\widetilde{\mathcal{D}}$ in (52) in the sense that it annihilates the gradient of the Hamiltonian [5].

5 Connection with the Modified Dispersive Water Wave Equation:

It is well known that the Harry Dym equation can be transformed to the MKdV equation under a hodograph transformation [16]. Similarly, the generalized Harry Dym equation can be transformed to two coupled MKdV systems under a hodograph transformation [18]. Such two component coupled MKdV systems have already been classified. In this section, we will study the connection of our system of equations with other equations. Let us note that if we set $h = 0$ in (37) (which would correspond to setting $W = 0$ for the supersymmetric generalized Harry Dym equation) the system of equations

takes the form

$$
\begin{aligned}
f_\tau &= \left(2f_{yy} - f^3 - 6g_y g - 3g^2 f\right)_y \\
g_\tau &= \left(2g_{yy} - 5g^3 + 6gf_y - 3gf^2\right)_y .
\end{aligned} \tag{53}
$$

This is a system of two coupled MKdV equations and under this reduction the Hamiltonian structure can be obtained from the Dirac reduction of the operator $\widetilde{\mathcal{D}}$ as discussed earlier and we obtain the two by two matrix $\widetilde{\mathcal{D}}$ with elements

$$
\begin{aligned}
\widetilde{\mathcal{D}}^{(1,1)} &= -2f_y\partial^{-1}f_y + \partial f^2 + f^2\partial - 2\partial^3, \\
\widetilde{\mathcal{D}}^{(1,2)} &= -2f_y\partial^{-1}g_y + 2\partial(\partial g + g\partial) + 2\partial g f + 2g\partial f, \\
\widetilde{\mathcal{D}}^{(2,2)} &= -2g_y\partial^{-1}g_y + \partial(f^2 + 4g^2 - 2f_y) + (f^2 + 4g^2 - 2f_y)\partial - 2\partial^3.
\end{aligned} \tag{54}
$$

We note that, as in the previous case, $\int \mathrm{d}y\, f$ is conserved and defines the Casimir of the Hamiltonian structure. However, in the present case, $\tilde{H} = \int \mathrm{d}y\, g$ is also conserved and can be used to construct a new system of equations

$$
\begin{pmatrix} f \\ g \end{pmatrix}_\tau = \widetilde{\mathcal{D}} \begin{pmatrix} \frac{\delta \tilde{H}}{\delta f} \\ \frac{\delta \tilde{H}}{\delta g} \end{pmatrix} = \begin{pmatrix} 2(g_y + gf)_y \\ (-2f_y + 3g^2 + f^2)_y \end{pmatrix} . \tag{55}
$$

Under a change of variables

$$
f = (a+b), \quad g = i(a-b), \tag{56}
$$

the system of equations (53) takes the form

$$
\begin{aligned}
a_\tau &= (a_{yy} + 3aa_y - 3ba_y + a^3 - 6a^2 b + 3ab^2)_y, \\
b_\tau &= (b_{yy} + 3bb_y - 3ab_y + b^3 - 6b^2 a + 3ba^2)_y,
\end{aligned} \tag{57}
$$

which has been studied earlier by Sakovich [18] and Foursov [19]. On the other hand, under this change of variables (56), the new system of equations (55) has the form

$$
\begin{aligned}
a_\tau &= 2i(a_y + a^2 - 2ab)_y, \\
b_\tau &= 2i(-b_y - b^2 + 2ab)_y.
\end{aligned} \tag{58}
$$

Theses constitute the system of coupled Burgers equations [2] and in the following we will neglect the overall factor $2i$.

Both of the system of equations (57) and (58) are Hamiltonian with the Hamiltonian structure (it is the structure (54) in the new variables)

$$\begin{aligned}
\widetilde{\mathcal{D}}_0^{(1,1)} &= -2a_y\partial^{-1}a_y + \partial(a_y + 2a^2 - 2ab) + (a_y + 2a^2 - 2ab)\partial, \\
\widetilde{\mathcal{D}}_0^{(1,2)} &= -2a_y\partial^{-1}b_y - \partial^3 - \partial(\partial a - 2ab + a^2 + a\partial) + \\
&\quad (\partial b + 2ab - b^2 + b\partial)\partial + 2b\partial a, \\
\mathcal{D}_0^{(2,2)} &= -2b_y\partial^{-1}b_y + \partial(b_y + 2b^2 - 2ab) + (b_y + 2b^2 - 2ab)\partial. \qquad (59)
\end{aligned}$$

It is already known that the systems of equations (57) and (58) are bi-Hamiltonian [2, 21] with Hamiltonian structures $\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ which we describe below. However we find that, in fact, both these systems of equations are tri-Hamiltonian systems much like the two boson equation. For example, we can write the system of equations (58) as

$$\begin{pmatrix} a \\ b \end{pmatrix}_\tau = \widetilde{\mathcal{D}}_0 \begin{pmatrix} \frac{\delta \tilde{H}_0}{\delta a} \\ \frac{\delta \tilde{H}_0}{\delta b} \end{pmatrix} = \widetilde{\mathcal{D}}_1 \begin{pmatrix} \frac{\delta \tilde{H}_1}{\delta a} \\ \frac{\delta \tilde{H}_1}{\delta b} \end{pmatrix} = \widetilde{\mathcal{D}}_2 \begin{pmatrix} \frac{\delta \tilde{H}_2}{\delta a} \\ \frac{\delta \tilde{H}_2}{\delta b} \end{pmatrix}, \qquad (60)$$

where

$$\tilde{H}_0 = \int \mathrm{d}y\,(a-b), \quad \tilde{H}_1 = \int \mathrm{d}y\,ab, \quad \tilde{H}_2 = \int \mathrm{d}y\,\left(a^2 b - ab^2 + a_y b\right).$$

The other two Hamiltonian structures have been studied earlier to have the forms

$$\begin{aligned}
\widetilde{\mathcal{D}}_1 &= \begin{pmatrix} -2a\partial - a_y & \partial^2 + (a-b)\partial + a_y \\ -\partial^2 + (a-b)\partial - b_y & 2b\partial + b_y \end{pmatrix}, \\
\widetilde{\mathcal{D}}_2 &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \qquad (61)
\end{aligned}$$

Similarly, we can write the system of equations (57) as

$$\begin{pmatrix} a \\ b \end{pmatrix}_\tau = \widetilde{\mathcal{D}}_0 \begin{pmatrix} \frac{\delta \tilde{H}_1}{\delta a} \\ \frac{\delta \tilde{H}_1}{\delta b} \end{pmatrix} = \widetilde{\mathcal{D}}_1 \begin{pmatrix} \frac{\delta \tilde{H}_2}{\delta a} \\ \frac{\delta \tilde{H}_2}{\delta b} \end{pmatrix} = \widetilde{\mathcal{D}}_2 \begin{pmatrix} \frac{\delta \tilde{H}_3}{\delta a} \\ \frac{\delta \tilde{H}_3}{\delta b} \end{pmatrix}, \qquad (62)$$

where

$$\tilde{H}_3 = \frac{1}{2}\int \mathrm{d}y(2b^3a - 3b^2a_y - 6b^2a + 2ba_{yy} + 6ba_ya + 2ba^3). \qquad (63)$$

A careful analysis shows that the new Hamiltonian operator $\widetilde{\mathcal{D}}_0$ can, in fact, be written as

$$\widetilde{\mathcal{D}}_0 = -\widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2^{-1}\widetilde{\mathcal{D}}_1. \qquad (64)$$

This shows that the three Hamiltonian structures are related through the recursion operator

$$\mathcal{R} = \widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2^{-1},$$

as

$$\widetilde{\mathcal{D}}_1 = \mathcal{R}\widetilde{\mathcal{D}}_2, \quad \widetilde{\mathcal{D}}_0 = -\mathcal{R}^2\widetilde{\mathcal{D}}_2.$$

Consequently, these define compatible Hamiltonian structures and the associated Nijenhuis torsion tensor vanishes [20].

To close this section we point out that the system of equations (58) can be mapped to the modified dispersive water wave equations [2] under the transformation

$$a \rightarrow -v, \quad b \rightarrow w - v, \quad \tau \rightarrow -\frac{\tau}{2}. \qquad (65)$$

Namely, under this transformation (58) goes into

$$\begin{aligned} v_\tau &= \frac{1}{2}\big(- v_y + 2vw - v^2\big)_y, \\ w_\tau &= \frac{1}{2}\big(w_x - 2v_y - 2v^2 + 2vw + w^2\big)_y, \end{aligned} \qquad (66)$$

which corresponds to the modified dispersive water wave equation.

6 Dispersionless Limit:

Given a dispersive system, one can obtain the dispersionless limit by taking the long wavelength limit [22]. In this limit the three component system obtained in (51) takes the form

$$\begin{aligned} f_\tau &= \Big(- f^3 - 3g^2f + 3h^2f\Big)_y, \\ g_\tau &= \Big(- 5g^3 - 3gf^2 + 3h^2g\Big)_y + 6h_yhg, \\ h_\tau &= \Big(+ 5h^3 - 3hf^2 - 3hg^2\Big)_y - 6hg_yg. \end{aligned} \qquad (67)$$

This can be written in the Hamiltonian form with the dispersionless limit of the Hamiltonian structure $\widetilde{\mathcal{D}}$ in (52)

$$\begin{aligned}
\widetilde{\mathcal{D}}^{(1,1)} &= -2f_y\partial^{-1}f_y + \partial f^2 + f^2\partial, \\
\mathcal{D}^{(1,2)} &= -2f_y\partial^{-1}g_y + 2\partial gf + 2g\partial f, \\
\widetilde{\mathcal{D}}^{(1,3)} &= -2f_y\partial^{-1}h_y + 2\partial hf + 2h\partial f, \\
\widetilde{\mathcal{D}}^{(2,2)} &= -2g_y\partial^{-1}g_y + \partial(f^2+4g^2) + (f^2+4g^2)\partial, \\
\widetilde{\mathcal{D}}^{(2,3)} &= -2g_y\partial^{-1}h_y + 4(\partial hg + hg\partial), \\
\widetilde{\mathcal{D}}^{(3,3)} &= -2h_y\partial^{-1}h_y + \partial(4h^2-f^2) + (4h^2-f^2)\partial, \qquad (68)
\end{aligned}$$

and the Hamiltonian $H = \int \mathrm{d}y\,(-f^2 - g^2 + h^2)$.

If we now define the new variables

$$F = \frac{3}{4}f^2 - \frac{3}{4}G, \quad G = -3g^2 + H, \quad H = 3h^2, \qquad (69)$$

then in these variables the dynamical equations can be written as

$$\begin{aligned}
F_\tau &= \Big((F^2 - \frac{3}{4}G^2 - 2GF)_y + G_yF\Big), \\
G_\tau &= \Big((G^2 + 2GF)_y + 2GF_y\Big), \\
H_\tau &= \Big((2HF + HG)_y + 2HF_y\Big). \qquad (70)
\end{aligned}$$

The Hamiltonian structure correspondingly takes the form

$$\begin{aligned}
\widetilde{\mathcal{D}}^{(1,1)} &= \partial(4F^2 - \frac{9}{2}G^2 - 15GF) + (4F^2 - \frac{9}{2}G^2 - 15GF)\partial - 2F_y\partial^{-1}F_y, \\
\widetilde{\mathcal{D}}^{(1,2)} &= \partial(3G^2 + 20FG) + (3G^2 + 20FG)\partial + 4FG_y + 3G_yG - 2F_y\partial^{-1}G_y, \\
\widetilde{\mathcal{D}}^{(1,3)} &= 20\partial HF + (20HF + 6HG)\partial + 4FH_y + 9HG_y - 2F_y\partial^{-1}H_y, \\
\widetilde{\mathcal{D}}^{(2,2)} &= \partial(4G^2 - 16GF) + (4G^2 - 16GF)\partial - 2G_y\partial^{-1}G_y, \\
\widetilde{\mathcal{D}}^{(2,3)} &= -16(\partial HF + HF\partial) - 8HG\partial - 4HG_y + 12H_yG_y - 2G_y\partial^{-1}H_y, \\
\widetilde{\mathcal{D}}^{(3,3)} &= \partial(16HF + 12HG + 16H^2) + (16HF + 12HG + 16H^2)\partial \\
&\quad -2H_y\partial^{-1}H_y. \qquad (71)
\end{aligned}$$

As we see, the first two equations do not depend on H and the third equation allows us to set $H = 0$ consistently. In this case, with a redefinition of variables $f = 3a - 2b$ and $g = 4(b - a)$, the first two equations reduce to a much simpler system, namely,

$$
\begin{aligned}
a_\tau &= \Big(5a^2 - 4ab\Big)_y,\\
b_\tau &= \Big(-2b^2 + 2ab\Big)_y + 2ba_y.
\end{aligned}
\tag{72}
$$

In this case, the Dirac reduction of the Hamiltonian structure becomes singular. Nevertheless, we have explicitly verified, using the method of prolongation [23], that the Hamiltonian structure

$$
\begin{aligned}
\widetilde{\mathcal{D}}_0^{(1,1)} &= 4\partial(4a^2 - 3ab) + 4(4a^2 - 3ab)\partial - 2a_y\partial^{-1}a_y,\\
\widetilde{\mathcal{D}}_0^{(1,2)} &= 4\partial(2ab - b^2) + 4(2ab - b^2)\partial - 4ba_y - 4b_y b - 2a_y\partial^{-1}b_y,\\
\widetilde{\mathcal{D}}_0^{(2,2)} &= 4(\partial b^2 + b^2\partial) - 2b_y\partial^{-1}b_y,
\end{aligned}
\tag{73}
$$

satisfies Jacobi identity and generates the dynamical equations as a Hamiltonian system with the Hamiltonian $\tilde{H}_0 = \frac{1}{3}\int \mathrm{d}y\, a$. Furthermore, this new system of equations (72) is, in fact, bi-Hamiltonian

$$
\begin{pmatrix} a \\ b \end{pmatrix}_\tau = \widetilde{\mathcal{D}}_0 \begin{pmatrix} \frac{\delta \tilde{H}_0}{\delta a} \\ \frac{\delta \tilde{H}_0}{\delta b} \end{pmatrix} = \widetilde{\mathcal{D}}_1 \begin{pmatrix} \frac{\delta \tilde{H}_1}{\delta a} \\ \frac{\delta \tilde{H}_1}{\delta b} \end{pmatrix},
\tag{74}
$$

where $\tilde{H}_1 = \int \mathrm{d}y\,(20a^2 - 16ab)$ and

$$
\widetilde{\mathcal{D}}_1 = \frac{1}{12}\begin{pmatrix} \partial a + a\partial & \partial b + b\partial \\ \partial b + b\partial & \partial b + b\partial \end{pmatrix}.
\tag{75}
$$

We can now construct the recursion operator $R = \widetilde{\mathcal{D}}^{-1}\widetilde{\mathcal{D}}_0$ and obtain the infinite series of conserved charges. The first few have the explicit forms

$$
\begin{aligned}
\tilde{H}_2 &= \int \mathrm{d}y\, a(-21a^2 - 8b^2 + 28ab),\\
\tilde{H}_3 &= \int \mathrm{d}y\, a(-429a^3 + 729a^2b - 432ab^2 + 64b^3),\\
\tilde{H}_4 &= \int \mathrm{d}y\, a(-2431a^4 + 5720a^3b - 4576a^2b^2 + 1408ab^3 - 128b^4).
\end{aligned}
\tag{76}
$$

In addition to these polynomial conserved charges, we can also construct a nonpolynomial series of conserved charges using known techniques in hydrodynamics, which we explain below.

Let us note that

$$\hat{H}_1 = \int \mathrm{d}y \sqrt{\frac{a}{b} - 1}, \quad \hat{H}_2 = \int \mathrm{d}y\, a\sqrt{a-b} \tag{77}$$

also define conserved quantities for the system. These charges have been constructed using the analog of the Tricomi equation used in the theory of polytropic gas dynamics [24]. Namely, if a conserved density H ($\hat{H} = \int \mathrm{d}y\, H$) depends on a and b only, it satisfies $H_\tau = G_y$ where G depends only on a and b. Using (72), this can be shown to lead to the following analog of the Tricomi equation

$$4aH_{a,b} + 2aH_{aa} + 2bH_{bb} + H_b = 0, \tag{78}$$

where the subscripts denote derivatives with respect to the particular variable. The conserved quantities H_i in (77) can be obtained as particular solutions of the Tricomi equation. We note that if we scale $H \to \sqrt{(a-b)}H$ then equation (78) transforms to

$$4aH_{a,b} + 2aH_{aa} + 2bH_{bb} + 3H_b = 0. \tag{79}$$

The general solutions of equations (78) and (79) can be written as

$$H_n = \sum_{k=0}^{n-1} \lambda_{k,n} a^{n-k} b^k, \tag{80}$$

where

$$\lambda_{k,n} = -2\lambda_{k-1,n}\frac{(n-k+1)(n-k)}{4k(n-k) + 2k(k-1) + zk} \tag{81}$$

for $k > 0$ with $\lambda_{0,n} = 0$ and $z = 1$ for (78) while $z = 3$ for (79). In the first case, we have polynomial charges the second leads to nonpolynomial charges (because of the factor $\sqrt{a-b}$).

If we further change the variable b to $b = 3c^2$, then equation (72) has the conservative form

$$\begin{aligned} a_\tau &= \Big(5a^2 - 12ac^2\Big)_y, \\ c_\tau &= \Big(-4c^3 + 2ac\Big)_y, \end{aligned} \tag{82}$$

which can be written in the bi-Hamiltonian form

$$\left(\begin{array}{c} a \\ c \end{array}\right)_{\tau} = \mathcal{D}_0 \left(\begin{array}{c} \frac{\delta H_0}{\delta a} \\ \frac{\delta H_0}{\delta b} \end{array}\right) = \mathcal{D}_1 \left(\begin{array}{c} \frac{\delta H_1}{\delta a} \\ \frac{\delta H_1}{\delta b} \end{array}\right), \tag{83}$$

where

$$\mathcal{D}_1 = \frac{1}{12}\left(\begin{array}{cc} \partial a + a\partial & c\partial \\ \partial c & \frac{1}{6}\partial \end{array}\right), \tag{84}$$

while $\mathcal{D}_0$ has the matrix elements

$$\begin{aligned}
\mathcal{D}_0^{(1,1)} &= 4\partial(4a^2 - 9ac^2) + 4(4a^2 - 9ac^2)\partial - 2a_y\partial^{-1}a_y, \\
\mathcal{D}_0^{(1,2)} &= -4(3c^3 - 2ca)\partial + 2ca_y - 2a_y\partial^{-1}c_y, \\
\mathcal{D}_0(2,2) &= \partial c^2 + c^2\partial - 2c_y\partial^{-1}c_y,
\end{aligned} \tag{85}$$

and $H_0 = \frac{1}{3}\int \mathrm{d}y\, a, H_1 = \int \mathrm{d}y\,(20a^2 - 48ac^2)$.

Finally let us comment on the different possible reductions of the system of equations (67). We note that when $h = 0$, we have

$$\begin{aligned}
f_\tau &= \Big(-f^3 - 3g^2 f\Big)_y, \\
g_\tau &= \Big(-5g^3 - 3gf^2\Big)_y,
\end{aligned} \tag{86}$$

and it corresponds to the dispersionless limit of equation (53). We will not discuss this further. If we set $f = 0$, then (69) leads to $F = \frac{9}{4}(g^2 - h^2), G = -3(g^2 - h^2), H = 3h^2$ and we have the system of equations

$$\begin{aligned}
G_\tau &= -\frac{5}{4}(G^2)_y, \\
H_\tau &= -\frac{1}{2}(HG)_y + HG_y,
\end{aligned} \tag{87}$$

where the first equation is decoupled.

7 Conclusion:

In this paper, we have constructed the two component supersymmetric generalized Harry Dym equation which is integrable and have studied various

properties of this model in the bosonic limit. In particular, we find a new integrable model in the bosonic limit which under a hodograph transformation maps to a system of three coupled MKdV equations. We have shown how the Hamiltonian structure transforms under a hodograph transformation and studied the properties of the system under a further reduction to a two component system. We find a third Hamiltonian structure for this system making this a genuinely tri-Hamiltonian system (it was known earlier to be a bi-Hamiltonian system). We have clarified the connection of this sytem to the modified dispersive water wave equation and have studied various properties of our model in the dispersionless limit.

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References

[1] M. D. Kruskal, Lecture Notes in Physics **38** p. 310 (Springer 1975).

[2] M. Błaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Springer-Verlag 1998.

[3] J. K. Hunter and Y. Zheng, Physica **D79** (1994) 361.

[4] R. Camasa and D. Holm, Phys. Rev. Lett. **71** (1993) 1661.

[5] J. C. Brunelli and G. A. T. F. de Costa, J. Math. Phys. **43** (2002) 6116.

[6] H. Dai and M. Pavlov, J. Phys. Soc. Japan **67** (1998) 3655.

[7] J. C. Brunelli, A. Das and Z. Popowicz, *Deformed Harry Dym and Hunter-Zheng Equations*, nlin.SI/0307043, to be published in J. Math. Phys.

[8] J. C. Brunelli, A. Das and Z. Popowicz, J. Math. Phys. **44** (2003) 4756.

[9] Q. P. Liu, J. Phys. **28A** (1995) L254.

[10] Z. Popowicz, Phys. Lett. **A317** (2003) 260.

[11] Y. Manin and R. Radul, Comm. Math. Phys. **158** (1985) 267.

[12] C. A. Laberge and P. Mathieu, Phys. Lett. **215B** (1988) 718.

[13] Z. Popowicz, J. Phys. **A29** (1996) 1281.

[14] W. Oevel and Z. Popowicz, Comm. Math. Phys. **139** (1991) 441.

[15] F. Delduc, L. Gallot and E. Ivanov, Phys. Lett. **B396** (1997) 122.

[16] N. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordecht 1985.

[17] A. Das and J. C. Brunelli, Int. J. Mod. Phys. **A10** (1995) 4563.

[18] S. Sakovich, *On bosonic limits of two recent supersymmetric extensions of the Harry Dym hierarchy*, nlin.SI/0310039.

[19] M. Foursov, Inverse Problems, **16** (2000) 259.

[20] A. Das and W.J. Huang, J. Math. Phys. **31** (1990) 2603.

[21] S. Sakovich, Phys. Lett. **A321** (2004) 252.

[22] V. E. Zakharov, Funct. Anal. Appl. **14** (1980) 89.

[23] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. (Springer, Berlin, 1993).

[24] M. Pavlov and Z. Popowicz, J. Phys. **A36** (2003) 8463.